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Spectral Distributions and Some Cauchy Problems

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Abstract—In this paper we give a simple representation of the solution of the Cauchy problem when the operator admits a spectral distribution. First we apply this to the Schrödinger operator on $\mathbb{R}^N$ with and without potential, and then on a bounded domain. In this case we give the expression of the associated spectral distribution. A second application is that for the Dirac operator.

Keywords—Functional calculus, Group of operators, Spectral distributions, Schrödinger operator, Dirac operator.

1. INTRODUCTION

A spectral distribution is an operator valued distribution defined on test functions (see [1,2]) that is extended naturally to C^∞ functions. The authors show that this symbolic calculus is admissible.

An important class of spectral distribution is that of order k, i.e., when the distribution could be extended continuously to the space $\check{\mathcal{T}}_k$, space of functions f such that $t^k \mathcal{F} f^{(k)} \in L^1(\mathbb{R})$. A generalisation of Stone's theorem is then proved, which states that A generates a k-times integrated group $(G(t))$ verifying $\|G(t)\| \le C|t|^k$ if and only if $B = iA$ admits a spectral distribution of order k.

The motivation of semigroup theory is to solve the following problem:

$$\frac{d}{dt}u(t) = iBu(t), \qquad u(0) = x. \tag{1}$$

When iB generates a C_0-semigroup $(G(t))$, the solution of (1) is then given by $u(t) = G(t)x$. In the case of integrated semigroup the expression is more complicated (see [3–5]). In this paper we will give a simple representation of the solution.

An extensive number of applications has been given in [6]. In particular they showed that the Schrödinger operator $A = i\Delta$ in $L^p(\mathbb{R}^N)$ with domain $W^{2,p}(\mathbb{R}^N)$ verifies the condition of the generalized Stone's theorem and then iA admits a spectral distribution of order $k \ge N|1/p - 1/2|$.

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In bounded domain, recently, using Sobolev imbeddings and an interpolation result for integrated semigroups, Arendt [7] showed that the Schrödinger operator on $L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$, generates a k-times integrated group for $k \geq (N/2)|1/p - 1/2|$. In Section 3 we give two ways to show that iA admits a spectral distribution of the same order k, the first based on Arendt's method, and the second is direct and explicit. Indeed we show how to construct the associated spectral distribution; nevertheless we lose in order of regularity.

A second application is given in Section 4 for the Dirac operator (see [8,9]). In [9], the author shows that the Dirac operator generates an α-times integrated group for $\alpha > 2|1/2 - 1/p| + 1$. Here we show that it generates a k-times integrated group of type $O(1 + |t|^k)$; hence it admits a $\check{\mathcal{T}}_k$ $\mathcal{D}^{-k}$-regularized spectral distribution for $k \geq 2|1/2 - 1/p|$.

2. SPECTRAL DISTRIBUTION AND THE CAUCHY PROBLEM

A *spectral distribution* is a linear algebra homomorphism $\mathcal{E}$ from $\mathcal{D} \equiv C_0^\infty(\mathbb{R})$ into $\mathcal{L}(X)$ verifying $\lim_{n\to\infty} \mathcal{E}(\varphi_n) = I$, where $\varphi_n(t) \equiv \varphi(t/n)$, with $\varphi \in \mathcal{D}$ such that $\varphi(0) = 1$. For every C^∞ function f, $\mathcal{E}(f)$ is defined as the limit of $\mathcal{E}(\varphi_n f)$ as $n \to \infty$.

The spectral distribution $\mathcal{E}$ is said to be *of degree* ℓ if it can be extended as a linear continuous mapping on $\check{\mathcal{T}}_\ell \equiv \mathcal{F}^{-1}\mathcal{T}_\ell$ equipped with the norm $\Pi_\ell(f) \equiv p_\ell(\mathcal{F}f)$, where $\mathcal{F}$ is Fourier transformation and $\mathcal{T}_\ell$ the completion of $\mathcal{D}$ for the norm p_ℓ defined by $p_\ell(\varphi) \equiv \sum_{k=0}^{\ell} \|t^k \varphi^{(k)}\|_{L^1}$.

LEMMA 2.1.

(1) *For all $\varphi \in \mathcal{D}$, $s > 0$, define $\varphi_s(t) \equiv s^{-1}\varphi(t/s)$. Then for any positive integer $k > 0$, $p_k(\varphi_s)$ is independent of s. In the same way $\Pi_k(\varphi_n)$ is independent of n, where $\varphi_n(t) \equiv \varphi(t/n)$.*

(2) *For any integer $\ell \geq 1$, $\mu \in \mathbb{C}$, $\operatorname{Re}\mu > 0$, the function $t \longmapsto H(t)t^{\ell-1}e^{-\mu t}$ belongs to $\mathcal{T}$, where $H(t)$ is the Heaviside function.*

(3) *For any integer $\ell \geq 1$, $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, the function $t \longmapsto (\lambda - t)^{-\ell} \in \check{\mathcal{T}}$.*

(4) *For any integer $\ell \geq 2$, $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \neq 0$, and $s \in \mathbb{R}$, the function $G_s : t \longmapsto t(\lambda - t)^{-\ell}e^{ist}$ belongs to $\check{\mathcal{T}}$, and $\Pi_k(G_s) \leq C(1 + |s|^k)$.*

PROOF.

(1) Direct calculation.

(2) For $\ell = 1$ see [2]. If not, let $m = \min\{k, \ell - 1\}$, and we have

$$p_k\left(H(\tau)\tau^{\ell-1}e^{i\lambda\tau}\right) \leq \sum_{j=0}^{m} C_k^j \frac{(\ell-1)!}{(k-j-1)!} \left\|\mu^{k-j}H(\tau)\tau^{k+\ell-j-1}e^{-\mu t}\right\|_{L^1}$$

(3) It suffices to notice that the Fourier transform of the function $t \longmapsto (\lambda - t)^{-\ell}$ is the function $\tau \longmapsto 1/\ell! H(\varepsilon\tau)(\varepsilon\tau)^{\ell-1}e^{i\lambda\varepsilon\tau}$, where $\varepsilon = +1$ if $\operatorname{Im}\lambda > 0$ and $\varepsilon = -1$ elsewhere.

(4) Write

$$\frac{t}{(\lambda - t)^\ell} = \frac{\lambda}{(\lambda - t)^\ell} - \frac{1}{(\lambda - t)^{\ell-1}},$$

and for $\ell - 1 \geq 1$ each one of them is in $\check{\mathcal{T}}$. Now note that if f verifies $t^i \mathcal{F}f^{(k)} \in L^1$ for every $i \leq k$, then the function $g : t \longmapsto e^{ist}f(t)$ belongs to $\check{\mathcal{T}}$ since $\mathcal{F}g(\tau) = \mathcal{F}f(\tau - s/2\pi)$ and by a simple change of variable we get $\Pi_k(g) = \|t^k \mathcal{F}f^{(k)}(t - s/2\pi)\|_{L^1} = \|(t + s/2\pi)^k \mathcal{F}f^{(k)}(t)\|_{L^1} \leq C(1 + |s|^k)$. ∎

THEOREM 2.2. *Let B be the momentum of a spectral distribution $\mathcal{E}$ of degree k. For every $x \in D(B^{k+1})$, $s \in \mathbb{R}$, $u(s) = \mathcal{E}(t \longmapsto (\lambda - t)^{-k-1}e^{ist})(\lambda - B)^{k+1}x$ is the unique solution of the Cauchy problem (1), and this solution is polynomially bounded, i.e., $\|u(s)\| \leq C(1 + |s|^k)\|(\lambda - B)^{k+1}x\|$ for every $s \in \mathbb{R}$.*

Note that $u(s)$ can be written in the form $u(s) = (\lambda - B)^{-k-1}e^{isB}(\lambda - B)^{k+1}$ and this product is not commutative since e^{isB} is not a bounded operator (see [1, Section 4]).

PROOF. First of all we show that for every $y \in X$, $u = \mathcal{E}(t \longmapsto (\lambda - t)^{-k-1}e^{ist})y \in D(B)$. Writing $B = \mathcal{E}(t)$ and by its definition, it is sufficient to show that $\lim_{n\to\infty} \mathcal{E}(t\varphi_n)u$ exists. We have $\mathcal{E}(t\varphi_n)u = \mathcal{E}(t\varphi_n)\mathcal{E}(t \longmapsto (\lambda - t)^{-k-1}e^{ist})y = \mathcal{E}(\varphi_n)\mathcal{E}(t \longmapsto t(\lambda - t)^{-k-1}e^{ist})y$; all these operators are bounded since these functions are in $\check{T}_k$ (see Lemma 2.1), so $\lim \mathcal{E}(t\varphi_n)u$ exists and $u \in D(B)$.

Now for $x \in X$ set $y = (\lambda - B)^{k+1}x$ and $u(s) = (\lambda - B)^{-k-1}e^{isB}(\lambda - B)^{k+1}$; by the last calculation and [1, Lemma 4.4] we have $\frac{d}{ds}u(s,x) = \mathcal{E}(t \longmapsto t(\lambda - t)^{-k-1}e^{ist})y = i\mathcal{E}(t)\mathcal{E}(t \longmapsto (\lambda - t)^{-k-1}e^{ist})y = iBu(s,x)$ and $u(0) = \mathcal{E}(t \longmapsto (\lambda - t)^{-k-1})y = x$. The growth inequality that verifies u is then a direct application of Lemma 2.1.(4).

For the uniqueness let $v(s)$ be another solution of (1), then using Lemma 2.3, we have $\frac{d}{ds}(\lambda - B)^{-k-1}e^{i(t-s)B}v(s)) = -iB(\lambda - B)^{-k-1}e^{i(t-s)B}v(s) + (\lambda - B)^{-k-1}e^{i(t-s)B}\frac{d}{ds}v(s) = 0$, so this function is constant and $(\lambda - B)^{-k-1}v(t) = (\lambda - B)^{-k-1}e^{itB}v(0) = (\lambda - B)^{-k-1}e^{itB}x$. Since $(\lambda - B)^{-k-1}$ is injective we get the desired result, $v(t) = u(t)$. ∎

3. SPECTRAL DISTRIBUTIONS AND INTERPOLATION

In [7], the authors give an interpolation theorem stating that each generator of integrated semigroup $(S(t))$ generates a C_0-semigroup $(U(t))$ on some subset G (see [7, Theorem 3.1 and Remark 3.2; 5]). In this section we will use those results to obtain spectral distributions.

Since the Schrödinger operator on L^2 generates a unitary group, it is natural to consider the following, here we use notations of [7].

PROPOSITION 3.1. *If the C_0-semigroup $\{U(t)\}$ is uniformly bounded on G, then the integrated semigroup $\{S(t)\}$ is of type $O(|t|^k)$, i.e., satisfies $\|S(t)\| \leq Ct^k$, for all $t \geq 0$, where C is a positive constant.*

PROOF. Taking $r \in \rho(A)$, $x \in X$, $y \equiv (r - A)^{-k}x \in D(A^k) = G$ (see [7, Remark 3.2]), we have

$$\begin{aligned} S(t)x &= (r - A)^k S(t)y \\ &= (r - A)^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} U(s)y\,ds \\ &= (r - A_G)^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} U(s)y\,ds \\ &= \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} (r - A_G)^k U(s)y\,ds, \end{aligned}$$

hence

$$\begin{aligned} \|S(t)x\| &\leq \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} C\left\|(r - A_G)^k U(s)y\right\| ds \\ &\leq \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} C\|U(s)y\|_G\,ds \\ &\leq \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} C\|y\|_G\,ds \\ &\leq C|t|^k\|y\|_G \\ &\leq M|t|^k\|x\|_E. \end{aligned}$$ ∎

Unfortunately, the converse is not true in general. However we have the following result.

PROPOSITION 3.2. *Let $(G(t))_{t\geq 0}$ be a k-times integrated semigroup verifying $\|G(t)\| \leq |p(t)|$, where p is a polynomial of degree $\leq k$. Then $G(t)$ verifies $\|G(t)(\lambda - A)^{-k})\| \leq Ct^k$, where C is a positive constant and $\lambda \in \rho(A)$.*

Note that this is natural since we have $G^{(i)}(0) = 0$ for every $i = 0, \dots, k-1$.

PROOF. Note that we can replace $|p(t)|$ by $M(1+t^k)$. Since it is obvious for $t \ge 1$, suppose that $t < 1$. For $x \in D(A^k)$, the application $t \longmapsto G(t)x$ is k-times differentiable vector valued function and

$$G(t)x = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} G^{(k)}(s)x\, ds.$$

Since $G^{(k)}(s)x = G(s)A^k x + \sum_{i=0}^{k-1}(s^i/i!)A^i x$, we have $\|G^{(k)}(s)x\| \le K(1+s^k)\|x\|_{D(A^k)}$, where $\|x\|_{D(A^k)} = \sum_{i=0}^{k-1} \|A^i x\|$. Therefore

$$\begin{aligned} \|G(t)x\| &\le K\|x\|_{D(A^k)} \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} \left(1+s^k\right) ds \\ &\le K\|x\|_{D(A^k)} \frac{t^k}{(k-1)!} \left(1+t^k\right) \le Kt^k \|x\|_{D(A^k)}. \end{aligned}$$

The uniform boundedness principle gives us the desired inequality. ∎

The following is a slight modification of [4, Theorem 2.4].

PROPOSITION 3.3. *Let A be the generator of a $(\lambda - A)^{-k}$-regularized semigroup $\{W(t)\}$ that is of type $O(1+t^k)$ for some λ. Suppose that $o \in \rho(A)$, then A generates a k-times integrated $\{S(t)\}$ semigroup of the same type.*

THEOREM 3.4. *Let A be as in Proposition 3.3. Then iA admits a $\check{T}_k$ $(w - H_p)^{-k}$-regularized spectral distribution.*

PROOF. Exactly as in [1], $\mathcal{E}(f) \equiv \int (-1)^k (\mathcal{F}f)^{(k)}(t) S(t)\, dt$, where $\{S(t)\}$ is as in the last proposition, define a regularized spectral distribution since, by Proposition 3.2, $S(t)$ verifies $\|S(t)(\lambda - A)^{-k})\| \le Ct^k$. ∎

4. EXAMPLES

4.1. Schrödinger Operator

4.1.A. Schrödinger operator on $L^p(\mathbb{R}^N)$

In [6], we showed that the Schrödinger operator without potential admits a spectral distribution for $k > N|1/2 - 1/p|$.

4.1.B. Schrödinger operator on bounded domain

With the notation of Arendt, let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, let Δ_p, $(1 \le p \le \infty)$ be the Laplace operator on $L^p(\Omega)$ with maximal domain, $A_p \equiv i\Delta_p$, A_0 and A_c are the parts of A_p on $C_0(\Omega)$ (continuous functions vanishing at infinity) and $C(\overline{\Omega})$, respectively.

The following is a direct application of [7, Theorems 4.2 and 4.3; 1, Theorem 3.4], and Proposition 3.1.

COROLLARY 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of $\mathbb{R}^N$, with boundary of class C^2. Then we have the following.*

(i) *If $N \le 3$, then iA_p $(1 \le p \le \infty)$ as well as iA_0 and iA_c admit a spectral distribution of order 1 in $L^p(\Omega)$, $C_0(\Omega)$, and $C(\overline{\Omega})$, respectively.*
(ii) *Let $1 < p < \infty$, if $k \ge (N/2)|1/2 - 1/p|$, then iA_p admits a spectral distribution of order k in $L^p(\Omega)$.*
(iii) *If $k > N/4$, then iA_0, iA_c, iA_∞, and iA_1 admit a spectral distribution of order k in $C_0(\Omega)$, $C(\overline{\Omega})$, $L^\infty(\Omega)$, and $L^1(\Omega)$, respectively.*

REMARK 4.2. Using Theorem 2.4, we can translate the last corollary in terms of solution of the problem $\frac{d}{dt}u(t) = Au(t)$, $u(0) = x$, to get a simple representation of the solution of the form $u(t) = (\lambda - iA)^{-k-1}e^{tA}(\lambda - iA)^{k+1}x$ where A is one of the Schrödinger operators as in Corollary 4.1 on the mentioned domain with the associated order k, and the solution $u(t)$ is polynomially bounded as in Theorem 2.2.

Direct construction of the spectral distribution of the Schrödinger operator on bounded domain

Consider the Schrödinger equation $\frac{\partial u}{\partial t} - i\Delta u = 0$, $x \in \Omega$ with the initial condition $u(0, x) = \phi(x)$, where Ω is a bounded open of $\mathbb{R}^N$ of class C^2, $\phi \in L^p(\Omega)$, $1 < p < \infty$. Here we will give the expression of the spectral distribution generated by this equation.

We know that the spectrum of $-\Delta$ is the set of an increasing sequence of positive eigenvalues (λ_n) which tends to infinity. Let (f_n) be the associated eigenvectors and let k be an integer such that $k \geq (N/2)|1/2 - 1/p|$. Now for every $\psi \in \tilde{\mathcal{T}}_k$ we define for $\phi \in L^p(\Omega)$

$$\begin{aligned}\mathcal{E}(\psi)\phi(x) &= \sum_n c_n(\phi) f_n(x) \int_{\mathbb{R}} e^{-i\lambda_n t}(-it)^k \hat{\psi}(t)\, dt \\ &= \sum_n c_n(\phi) f_n(x) \psi^{(k)}(-\lambda_n),\end{aligned}$$

where $\hat{\psi}$ is Fourier transform of ψ, $c_n(\phi)$ is the n^{th} Fourier coefficient, $c_n(\phi) = \int \phi \bar{f}_n$. Now since $|c_n(\phi)| \leq \|\phi\|_{L^p} \|\bar{f}_n\|_{L^q}$, where q is the conjugate of p, we have

$$\|\mathcal{E}(\psi)\phi\|_{L^p} \leq \sum_n |c_n(\phi)| \|f_n\|_{L^p} \left|\psi^{(k)}(-\lambda_n)\right| \leq \|\phi\|_{L^p} \sum_n \|f_n\|_{L^p} \|f_n\|_{L^q} \left|\psi^{(k)}(-\lambda_n)\right|,$$

using Sobolev imbeddings, and for $s \geq (N/2)|1/2 - 1/p|$, we have $\|f_n\|_{L^p} \leq C\|f_n\|_{H^{2s}} \leq C'(1 + \lambda_n^s)\|f_n\|_{L^2} = C'(1 + \lambda_n^s)$ and $\|f_n\|_{L^q} \leq K\|f_n\|_{L^2} = K$, where C, C', and K are constants. Hence we get

$$\begin{aligned}\|\mathcal{E}(\psi)\| &\leq M \sum_n \frac{1 + \lambda_n^s}{\lambda_n^k} \left|\lambda_n^k \psi^{(k)}(-\lambda_n)\right| \\ &\leq M \sum_n \frac{1 + \lambda_n^s}{\lambda_n^k} \left|\mathcal{F}\left(t^k \hat{\psi}^{(k)}(\lambda_n)\right)\right| \\ &\leq \pi_k(\psi) \sum_n \frac{1 + \lambda_n^s}{\lambda_n^k} \leq M'' \pi_k(\psi),\end{aligned}$$

since as $n \to \infty$, $\lambda_n \simeq n^{2/N}$, hence $(1 + \lambda_n^s)/(\lambda_n^k) \simeq (n^{2s/N})/(n^{2k/N})$, which means that we must have $2k/N > 2s/N + 1$, i.e., k must be $k > (N/2)|1/2 - 1/p| + N/2$.

Note that we have lose in order comparing with 4.1(ii).

4.1.C. Schrödinger operator with potential

Denote by K^N the *Kato class* of measurable functions on $\mathbb{R}^N$, as defined in [10, p. 453]. In [11], it is shown that, for $1 \leq p < \infty$, $V_+ \in K^N$, $V_- \in L^\infty(\mathbb{R}^N)$, $iH_p \equiv i\Delta - iV$ generates an $(w - H_p)^{-k}$-regularized group on $L^p(\mathbb{R}^N)$ that is of type $O(1 + |t|^k)$ for any integer $k > 2n|1/2 - 1/p|$; hence by Theorem 3.4, H_p admits a $\tilde{\mathcal{T}}_k$ $(w - H_p)^{-k}$-regularized spectral distribution.

4.2. Dirac Operator

Let

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_j \equiv \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \qquad (1 \leq j \leq 3),$$

then the Dirac operator is defined to be $\mathcal{D} \equiv \mathcal{A} + \mathcal{B}$, where

$$\mathcal{A} \equiv c \sum_{j=0}^{n} B_j \frac{\partial}{\partial x_j}, \qquad \mathcal{B} \equiv \mu c^2 \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

μ is the mass of the electron, c is the speed of light.

In [9, Theorem 3.2], the author showed that $\mathcal{D}$ defined on $L^p(\mathbb{R}^3)^4$ $(1 < p < \infty)$ with domain $D(A) \times D(A)$ generates a k-times integrated semigroup whenever $k > 2|1/2 - 1/p| + 1$. In the following, Theorem 4.3, we will show that the Dirac operator generates a k-times integrated group of type $O(1 + |t|^k)$ for $k \geq 2|1/2 - 1/p|$, hence $i\mathcal{D}$ admits a $\check{\mathcal{T}}_k$ $\mathcal{D}^{-k}$-regularized spectral distribution.

THEOREM 4.3. *The Dirac operator $\mathcal{D}$ generates a k-times integrated group of type $O(1 + |t|^k)$ for $k \geq 2|1/2 - 1/p|$, hence $i\mathcal{D}$ admits a $\check{\mathcal{T}}_k$ $\mathcal{D}^{-k}$-regularized spectral distribution.*

PROOF. Since $i\mathcal{A}$ admits a spectral distribution of order k (see [12]), then A generates $\{W(t)\}$, a $(\lambda - \mathcal{A})^{-k}$-regularized group of type $O(1 + |t|^k)$. And since $\mathcal{B}$ generates a unitary group $\{H(t)\}$, then $\mathcal{D}$ generates a $(\lambda - \mathcal{D})^{-k}$-regularized group of the same type. We get the desired result by applying Proposition 3.3. ∎

REMARK 4.4. In this case also, by using Theorem 2.4, we can translate the last corollary in terms of solution of the Cauchy problem as in 3.6 $u(t) = (\lambda - i\mathcal{D})^{-k-1} e^{t\mathcal{D}} (\lambda - i\mathcal{D})^{k+1} x$ where $\mathcal{D}$ is the Dirac operator and the solution $u(t)$ is polynomially bounded as in Theorem 2.4.

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